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Countable compacta of continuity for projective functions ☆

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ABSTRACT

We investigate connections between complexity of a function f from a Polish space X to a Polish space Y and complexity of the set $\widetilde{C}(f) = \{K \in \mathcal{K}(X);\ f\restriction_K$ is continuous and K has exactly one limit point$\}$, where $\mathcal{K}(X)$ denotes the space of all compact subsets of X equipped with the Vietoris topology. We prove that $\widetilde{C}(f)$ is Borel if and only if f is Borel. Similar results for projective classes are also presented.

1. Introduction

Let X be a Polish space. Denote the space of all compact subsets of X, which is equipped with the Vietoris topology, by $\mathcal{K}(X)$. At least since the important paper [4] it is well known that descriptive properties of families of compact sets and their set structure (like being σ-ideal or ideal) can interact in a nontrivial way (see [5] for a recent survey). These general results were applied in different parts of analysis, mainly in the theory of exceptional sets in harmonic analysis.

Let f be a function from X to a Polish space Y. We define

$$C(f) := \{K \in \mathcal{K}(X);\ f\restriction_K \text{ is continuous}\}.$$

F. Jordan [1,2] investigates relationships between descriptive properties of the function f and the ideal $C(f)$. Besides other results he showed that if f is Borel then f is a Baire class one function provided $C(f)$ is a $\boldsymbol{\Pi}^0_3$ subset of $\mathcal{K}(X)$. He also proved that if f is Borel then $C(f)$ is analytic if and only if f has $\boldsymbol{\Pi}^0_2$ graph.

V. Vlasák and M. Zelený [6] showed that the assumption of Borelness of f can be omitted. They also proved that $C(f)$ is analytic if and only if $C(f)$ is Borel, and gave the following characterization of Borel functions.

Theorem 1.1 (Det($\boldsymbol{\Delta}^1_2$)). *([6, Theorem 1.3]) Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then f is Borel if and only if $C(f)$ is a coanalytic subset of $\mathcal{K}(X)$.*

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In this note we investigate the collection

$$\widetilde{C}(f) := \left\{K \in \mathcal{K}(X);\ f\restriction_K \text{ is continuous and } K \text{ has exactly one limit point}\right\},$$

where $f : X \to Y$. Compacta with one limit point are in fact convergent sequences with limit point. So, we investigate continuity of functions on convergent sequences. But, by Heine theorem we have that continuity on convergent sequences characterizes continuity on general sets. We show that Borelness of f can be actually characterized by descriptive properties of $\widetilde{C}(f)$ assuming no determinacy axiom.

Theorem 1.2. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then the following are equivalent*:

(i) *$\widetilde{C}(f)$ is Borel,*
(ii) *$\widetilde{C}(f)$ is analytic,*
(iii) *f is Borel.*

One can show that $\widetilde{C}(f)$ is coanalytic if and only if $C(f)$ is coanalytic (see Lemma 3.9). Thus, Theorems 1.1 and 1.2 imply the next corollary giving a restriction on complexity of sets of the form $\widetilde{C}(f)$.

Corollary 1.3 *(*Det($\mathbf{\Delta}^1_2$)*). Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then $\widetilde{C}(f)$ is Borel if and only if $\widetilde{C}(f)$ is a coanalytic subset of $\mathcal{K}(X)$.*

We do not know whether the assumption on determinacy of $\mathbf{\Delta}^1_2$ games can be omitted in Corollary 1.3. In fact, the following two statements are equivalent in ZFC:

(i) $\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$ if and only if $\widetilde{C}(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$,
(ii) f is $\mathbf{\Delta}^1_n$-measurable if and only if $C(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$.

This equivalency simply follows from Theorem 2.1 and Lemma 3.9. So, if we proved Corollary 1.3 using a weaker axiom, then we would be able to prove Theorem 1.1 using the same axiom.

We also show several connections between a Baire class of a function f and descriptive properties of $\widetilde{C}(f)$.

We also study the property $C(f) = C(g)$. Clearly, $C(f) = C(g)$ if and only if $\widetilde{C}(f) = \widetilde{C}(g)$. We show that two Borel functions f and g with $C(f) = C(g)$ belong to the same Baire class. We also show that if f is Lebesgue measurable and $C(f) = C(g)$ then g is also Lebesgue measurable.

As for the notation and all needed definitions we refer to [3].

2. Results

Let us describe the main results of the paper. The following theorem provides a characterization of $\mathbf{\Delta}^1_n$-measurable functions. This is a generalization of Theorem 1.2 to projective classes, Theorem 1.2 is a special case of Theorem 2.1 for $n = 1$.

Theorem 2.1. *Let X, Y be Polish spaces, $n \in \mathbb{N}$ and $f : X \to Y$ be a function. Then the following are equivalent*:

(i) *$\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$,*
(ii) *$\widetilde{C}(f) \in \mathbf{\Sigma}^1_n(\mathcal{K}(X))$,*
(iii) *f is $\mathbf{\Delta}^1_n$-measurable.*

The next corollary is a more general version of Corollary 1.3.

Corollary 2.2 *(*Det($\mathbf{\Delta}^1_2$)*). Let X, Y be Polish spaces, $f : X \to Y$ be a function and $n \in \mathbb{N}$. Then $\widetilde{C}(f) \in \mathbf{\Delta}^1_n(\mathcal{K}(X))$ if and only if $\widetilde{C}(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$.*

Let X and Y be Polish spaces. The symbol $\mathcal{B}_\alpha(X, Y)$ stands for the set of all functions of Baire class α from X to Y. Now we define the classes $\mathcal{E}_\alpha$. These classes were studied by Jordan [2] who showed their interesting connections with Baire classes and collections $C(f)$.

Definition 2.3. Let X, Y be Polish spaces, $2 \leqslant \alpha < \omega_1$. We define $\mathcal{E}_\alpha(X, Y)$ as a collection of functions f such that for all $x \in X$ and W open neighborhood of $f(x)$ there exist $G \in \boldsymbol{\Pi}^0_\beta(X)$ with $\beta < \alpha$ and open sets $U \subset X$, $V \subset Y$ such that $x \in f^{-1}(V) \cap U$, $f^{-1}(V) \cap U \subset G$ and $f(G) \subset W$.

The following theorem shows us that if functions f and g satisfy $C(f)=C(g)$ then descriptive properties of f are very similar to descriptive properties of g.

Theorem 2.4. *Let X, Y be Polish spaces, $1 \leqslant \alpha < \omega_1$, $\mathcal{A}$ be a σ-algebra on X containing $\mathbf{\Delta}^1_1(X)$ and $f, g : X \to Y$ be functions with $C(f)=C(g)$.*

(i) *If $\alpha \geqslant 2$, then we have $f \in \mathcal{E}_\alpha(X,Y)$ if and only if $g \in \mathcal{E}_\alpha(X,Y)$.*
(ii) *We have $f \in \mathcal{B}_\alpha(X,Y)$ if and only if $g \in \mathcal{B}_\alpha(X,Y)$.*
(iii) *We have f is $\mathcal{A}$-measurable if and only if g is $\mathcal{A}$-measurable.*

Using (iii) from previous theorem we can simply prove the following two corollaries.

Corollary 2.5. *Let $n \in \mathbb{N}$, $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions and $C(f)=C(g)$. Then f is Lebesgue measurable if and only if g is Lebesgue measurable.*

Corollary 2.6. *Let X, Y be Polish spaces, $f, g : X \to Y$ be functions and $C(f)=C(g)$. Then f is Baire measurable if and only if g is Baire measurable.*

Jordan [1] shows that $f \in \mathcal{B}_1$ implies that $C(f)$ is $\boldsymbol{\Pi}^0_4$. He also shows that $C(f)$ is Borel if and only if f has $\boldsymbol{\Pi}^0_2$ graph. Thus, there exists $f \in \mathcal{B}_2$ such that $C(f)$ is not Borel. So, we cannot find any upper bound on descriptive quality of $C(f)$ for higher Baire classes. This is why we use collection $\widetilde{C}(f)$ instead of $C(f)$ to study higher Baire classes.

Theorem 2.7. *Let X, Y be Polish spaces, $1 \leqslant \alpha < \omega_1$, and $f : X \to Y$ be a function.*

(i) *If $\widetilde{C}(f) \in \boldsymbol{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$ and $\alpha \geqslant 2$, then $f \in \mathcal{E}_\alpha(X,Y)$.*
(ii) *If $f \in \mathcal{E}_{\alpha+1}(X,Y)$, then $\widetilde{C}(f) \in \boldsymbol{\Pi}^0_{\alpha+4}(\mathcal{K}(X))$.*
(iii) *If $f \in \mathcal{E}_\alpha(X,Y)$ and α is a limit ordinal, then $\widetilde{C}(f) \in \boldsymbol{\Pi}^0_{\alpha+1}(\mathcal{K}(X))$.*

Theorem 2.8. *Let X, Y be Polish spaces, $1 \leqslant \alpha < \omega_1$, and $f : X \to Y$ be a function.*

(i) *If $\widetilde{C}(f) \in \boldsymbol{\Pi}^0_{\alpha+2}(\mathcal{K}(X))$, then $f \in \mathcal{B}_\alpha(X,Y)$.*
(ii) *If $f \in \mathcal{B}_\alpha(X,Y)$, then $\widetilde{C}(f) \in \boldsymbol{\Pi}^0_{\alpha+5}(\mathcal{K}(X))$.*

3. Proofs

3.1. Notation

Let X and Y be Polish spaces and $f : X \to Y$ be a continuous function. Then the function $\widehat{f} : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\widehat{f}(K) = f(K)$. Let Γ be a collection of subsets of X. We define its dual collection $\check{\Gamma}$ by

$$\check{\Gamma} = \{A \subset X;\ X \setminus A \in \Gamma\}.$$

The symbol $\boldsymbol{d}_{CB}$ denotes the Cantor–Bendixson derivative. If $A \subset X$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of A. We also define collections $\mathcal{O}(A)$, $\mathcal{S}^n(A)$, $\mathcal{S}^{<\omega}(A)$ by

$$\mathcal{O}(A) = \bigl\{K \in \mathcal{K}(A);\ \operatorname{card}\bigl(\boldsymbol{d}_{CB}(K)\bigr) = 1\bigr\},$$
$$\mathcal{S}^n(A) = \bigl\{K \subset A;\ \operatorname{card}(K) \leqslant n\bigr\}, \quad n \in \mathbb{N},$$
$$\mathcal{S}^{<\omega}(A) = \bigl\{K \subset A;\ \operatorname{card}(K) < \omega\bigr\}.$$

The symbols π_X and π_Y denote the projections from $X \times Y$ to X and to Y respectively. The symbol $\mathcal{N}$ denotes the Baire space $\mathbb{N}^{\mathbb{N}}$. If $x \in X$, then $\mathcal{U}(x)$ denotes the family of all open neighborhoods of x.

3.2. Proof of Theorem 2.1

Lemma 3.1. *Let X be a Polish space. Then $\mathcal{O}(X) \in \boldsymbol{\Pi}^0_3(\mathcal{K}(X))$.*

Proof. Set $M := \{\{x\};\ x \in X\}$. Clearly, M is closed in $\mathcal{K}(X)$ and $\mathcal{O}(X) = \boldsymbol{d}_{CB}^{-1}(M)$. Since $\boldsymbol{d}_{CB} : \mathcal{K}(X) \to \mathcal{K}(X)$ is a Baire class two function (see [3, 24.9]), we have $\mathcal{O}(X) \in \boldsymbol{\Pi}^0_3(\mathcal{K}(X))$. $\square$

Lemma 3.2. *Let X be a Polish space, $n \in \mathbb{N}$, $3 \leqslant \alpha < \omega_1$ and $A \subset X$.*

(i) *If $A \in \boldsymbol{\Pi}^0_\alpha(X)$, then $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Pi}^0_\alpha(\mathcal{K}(X))$.*
(ii) *If $A \in \boldsymbol{\Delta}^1_n(X)$, then $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X))$.*

Proof. (i) Clearly, $\mathcal{S}^{<\omega}(X) = \bigcup_{n\in\mathbb{N}} \mathcal{S}^n(X)$ and $\mathcal{S}^n(X)$ is closed in $\mathcal{K}(X)$. So, we have $\mathcal{S}^{<\omega}(X) \in \boldsymbol{\Sigma}^0_2(\mathcal{K}(X))$. Let $F \subset X$ be closed. Then $\mathcal{S}^{<\omega}(F) = \mathcal{K}(F) \cap \mathcal{S}^{<\omega}(X)$. Thus we have $\mathcal{S}^{<\omega}(F) \in \boldsymbol{\Sigma}^0_2(\mathcal{K}(X))$. Let $B \in \boldsymbol{\Pi}^0_2(X)$. Then $\mathcal{S}^{<\omega}(B) = \mathcal{K}(B) \cap \mathcal{S}^{<\omega}(X)$. Thus we have $\mathcal{S}^{<\omega}(B) \in \boldsymbol{\Sigma}^0_3(\mathcal{K}(X))$, since $\mathcal{K}(B) \in \boldsymbol{\Pi}^0_2(\mathcal{K}(X))$. Let A_i, $i \in \mathbb{N}$, be arbitrary subsets of X. Clearly,

$$\mathcal{S}^{<\omega}\Bigl(\bigcap_{i\in\mathbb{N}} A_i\Bigr) = \bigcap_{i\in\mathbb{N}} \mathcal{S}^{<\omega}(A_i),$$
$$\mathcal{S}^{<\omega}\Bigl(\bigcup_{i\in\mathbb{N}} A_i\Bigr) = \bigcup_{k\in\mathbb{N}} \mathcal{S}^{<\omega}\Bigl(\bigcup_{i\leqslant k} A_i\Bigr).$$

For $\alpha \geqslant 3$ and $A \in \boldsymbol{\Pi}^0_\alpha(X)$ we have by transfinite induction that $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Pi}^0_\alpha(\mathcal{K}(X))$.

(ii) Assume $A \in \boldsymbol{\Pi}^1_n(X)$. Since classes $\boldsymbol{\Pi}^1_n$ are closed under coprojection and

$$\mathcal{S}^{<\omega}(A) = \bigl\{K \in \mathcal{S}^{<\omega}(X);\ \forall x \in X\colon x \notin K \vee x \in A\bigr\}$$

we have $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$.

Assume $n = 1$. So, A is a Borel set. Then A is coanalytic. So, $\mathcal{S}^{<\omega}(A)$ is coanalytic. The set A is also analytic. So, there exists a closed set $B \subset X \times \mathcal{N}$ such that $A = \pi_X(B)$. Clearly, $\mathcal{S}^{<\omega}(A) = \widehat{\pi_X}(\mathcal{S}^{<\omega}(B))$. So, $\mathcal{S}^{<\omega}(A)$ is analytic. Thus, $\mathcal{S}^{<\omega}(A)$ is Borel.

Now assume $n > 1$. Then $A \in \boldsymbol{\Pi}^1_n(X)$. So, $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. The set A also belongs to $\boldsymbol{\Sigma}^1_n(X)$. So, there exists a set $B \in \boldsymbol{\Pi}^1_{n-1}(X \times \mathcal{N})$ such that $A = \pi_X(B)$. Clearly, $\mathcal{S}^{<\omega}(A) = \widehat{\pi_X}(\mathcal{S}^{<\omega}(B))$. So, $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Sigma}^1_n(\mathcal{K}(X))$. Thus, $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X))$. □

Definition 3.3. Let X be a Polish space, $\rho \leqslant 1$ be a compatible metric on X and $v \geqslant s > 0$. Define $\Phi : \mathcal{O}(X) \to X$ and $\Lambda_{v,s,\rho} : \mathcal{O}(X) \to \mathcal{K}(X)$ by

$$\bigl\{\Phi(A)\bigr\} = \boldsymbol{d}_{CB}(A),$$
$$\Lambda_{v,s,\rho}(K) = K \cap P\bigl(\Phi(K), s, v\bigr),$$

where $P(x, s, v)$ is defined by

$$P(x, s, v) = \bigl\{y \in X;\ s \leqslant \rho(x, y) \leqslant v\bigr\}, \quad x \in X.$$

Lemma 3.4. *Let X be a Polish space. Let $\Phi : \mathcal{O}(X) \to X$ be as in Definition* 3.3. *Then Φ is a Baire class one function.*

Proof. Let U be an arbitrary open subset of X and $\rho \leqslant 1$ be a compatible metric on X. Set $F_n := \{x \in X;\ \operatorname{dist}(x, X \setminus U) \geqslant \frac{1}{n}\}$. Let $K \in \Phi^{-1}(U)$. Then $K \cap U$ is compact. So, $\operatorname{dist}(K \cap U, X \setminus U) > 0$. Thus we have

$$\Phi^{-1}(U) = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \bigl\{K \in \mathcal{O}(X);\ \operatorname{card}\bigl(K \cap (X \setminus F_n)\bigr) \leqslant i\bigr\}.$$

Clearly, $\{K \in \mathcal{O}(X);\ \operatorname{card}(K \cap (X \setminus F_n)) \leqslant i\}$ is closed. So, $\Phi^{-1}(U) \in \boldsymbol{\Sigma}^0_2(\mathcal{O}(X))$. □

Lemma 3.5. *Let X be a Polish space and ρ be a fixed compatible metric on X. Then $\Lambda_{v,s,\rho} \in \mathcal{B}_2(\mathcal{O}(X), \mathcal{K}(X))$.*

Proof. Let $\Phi : \mathcal{O}(X) \to X$ be as in Definition 3.3. By Lemma 3.4 Φ is a Baire one function from $\mathcal{O}(X) \to X$. Thus, it is enough to verify that $g : \mathcal{O}(X) \times X \to \mathcal{K}(X)$ defined by

$$g(K, x) = K \cap P(x, s, v)$$

is a Baire class one function. The sets of the form $\mathcal{K}(U)$ and $M_U := \{K \in \mathcal{K}(X);\ K \cap U \neq \emptyset\}$, where U is open in X, form a subbase of the Vietoris topology. Let $U \subset X$ be an arbitrary open set. There exist open sets $U_n \subset X$, $n \in \mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} U_n = X \setminus U$. Thus, $M_U = \mathcal{K}(X) \setminus \bigcap_{n\in\mathbb{N}} \mathcal{K}(U_n)$. So, it is enough to verify that $g^{-1}(\mathcal{K}(U))$ is open in $\mathcal{O}(X) \times X$.

Suppose towards a contradiction that there exist $(K_n, x_n) \notin g^{-1}(\mathcal{K}(U))$, $n \in \mathbb{N}$ and $(K, x) \in g^{-1}(\mathcal{K}(U))$ such that

$$(K_n, x_n) \to (K, x).$$

Thus for every $n \in \mathbb{N}$ there exists $z_n \in (K_n \cap P(x_n, s, v)) \setminus U$. Clearly, $K \cup \bigcup_{j\in\mathbb{N}} K_j$ is compact. Since $z_n \in K \cup \bigcup_{j\in\mathbb{N}} K_j$ there exist $z \in X$ and a subsequence $\{z_{n_i}\}$ converging to z. Thus, $z \in (K \cap P(x, s, v)) \setminus U$, a contradiction. □

Lemma 3.6. *Let X be a Polish space, $n \in \mathbb{N}$, $1 \leqslant \alpha < \omega_1$ and $A \subset X$.*

(i) *If $A \in \boldsymbol{\Pi}^0_\alpha(X)$, then $\mathcal{O}(A) \in \boldsymbol{\Pi}^0_{\alpha+2}(\mathcal{K}(X))$.*
(ii) *If $A \in \boldsymbol{\Delta}^1_n(X)$, then $\mathcal{O}(A) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X))$.*

Proof. Let $\Lambda_{v,s,\rho}, \Phi$ be as in Definition 3.3. By Lemma 3.5 we have $\Lambda_{v,s,\rho} \in \mathcal{B}_2(\mathcal{O}(X), \mathcal{K}(X))$. By Lemma 3.4 we have that Φ is a Baire one function. Clearly,

$$\mathcal{O}(A) = \big\{K \in \mathcal{O}(X);\ \Phi(K) \in A \wedge \forall s, v \in \mathbb{Q},\ v \geqslant s > 0\colon\ \Lambda_{v,s,\rho}(K) \subset A\big\}.$$

Thus we have

$$\mathcal{O}(A) = \big\{K \in \mathcal{O}(X);\ \Phi(K) \in A \wedge \forall s, v \in \mathbb{Q},\ v \geqslant s > 0\colon\ \Lambda_{v,s,\rho}(K) \in \mathcal{S}^{<\omega}(A)\big\}.$$

So,

$$\mathcal{O}(A) = \Phi^{-1}(A) \cap \bigcap_{s,v\in\mathbb{Q},\ v\geqslant s>0} \Lambda^{-1}_{v,s,\rho}\big(\mathcal{S}^{<\omega}(A)\big).$$

(i) Assume $\alpha \geqslant 3$. By Lemma 3.2(i) we have $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Pi}^0_\alpha(\mathcal{K}(X))$. Thus, we have $\mathcal{O}(A) \in \boldsymbol{\Pi}^0_{\alpha+2}(\mathcal{O}(X))$. Using Lemma 3.1 we are done.

On the other hand, if $\alpha < 3$ then $\mathcal{O}(A) = \mathcal{O}(X) \cap \mathcal{K}(A) \in \boldsymbol{\Pi}^0_3(\mathcal{K}(X))$.

(ii) By Lemma 3.2(ii) we have $\mathcal{S}^{<\omega}(A) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X))$. Since classes $\boldsymbol{\Delta}^1_n$ are closed under Borel preimages and countable intersections (see [3, Proposition 37.1]), we have $\mathcal{O}(A) \in \boldsymbol{\Delta}^1_n(\mathcal{O}(X))$. Using Lemma 3.1 we are done. □

Lemma 3.7. *Let X, Y be Polish spaces and $f : X \to Y$ be a function. Then*

$$\widetilde{C}(f) = \widehat{\pi_X}\big(\mathcal{O}\big(\mathrm{graph}(f)\big)\big).$$

Proof. The proof is similar to [6, Lemma 3.1]. □

Lemma 3.8. *Let X be a Polish space and $M \subset X$ be a countable set. Then*

$$\mathcal{S} := \big\{K \in \mathcal{K}(X);\ \overline{K \cap M} = K\big\} \in \boldsymbol{\Pi}^0_3\big(\mathcal{K}(X)\big).$$

Proof. Let $\mathcal{W}$ be a countable open base of X. Then for $A, B \subset X$ we have

$$\overline{A} = \overline{B} \quad\Leftrightarrow\quad \forall V \in \mathcal{W}\colon\quad (A \cap V = \emptyset \Leftrightarrow B \cap V = \emptyset).$$

Thus,

$$\mathcal{S} = \big\{K \in \mathcal{K}(X);\ \forall V \in \mathcal{W}\colon (K \cap V = \emptyset) \vee (\exists d \in V \cap M\colon d \in K)\big\}.$$

So, $\mathcal{S} \in \boldsymbol{\Pi}^0_3(\mathcal{K}(X))$ since M is countable. □

Proof of Theorem 2.1. (iii) ⇒ (i) According to [3, Exercise 37.3] we have that $\mathrm{graph}(f) \in \boldsymbol{\Delta}^1_n(X \times Y)$. By Lemma 3.6 we have $\mathcal{O}(\mathrm{graph}(f)) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X \times Y))$. By Lemma 3.7 we have $\widetilde{C}(f) = \widehat{\pi_X}(\mathcal{O}(\mathrm{graph}(f)))$. Thus $\widetilde{C}(f) \in \boldsymbol{\Sigma}^1_n(\mathcal{K}(X))$. By [6, Theorem 2.5(i)] we have that $C(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. Since $\widetilde{C}(f) = C(f) \cap \mathcal{O}(X)$ we have $\widetilde{C}(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. Thus, $\widetilde{C}(f) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X))$.

(i) ⇒ (ii) This implication is trivial.

(ii) ⇒ (iii) Let F be an arbitrary closed subset of Y. It is sufficient to prove that

$$f^{-1}(F) \in \boldsymbol{\Sigma}^1_n(X).$$

Find a set $D \subset X \times Y$ which is a countable and dense subset of

$$A := \mathrm{graph}(f) \cap (X \times F).$$

By Lemma 3.8 we have

$$\mathcal{V} := \big(\mathcal{S}^{<\omega}(X) \cup \widetilde{C}(f)\big) \cap \big\{K \in \mathcal{K}(X);\ \overline{K \cap \pi_X(D)} = K\big\} \in \boldsymbol{\Sigma}^1_n\big(\mathcal{K}(X)\big).$$

We prove that $f^{-1}(F) = \bigcup \mathcal{V}$, which implies that $f^{-1}(F) \in \boldsymbol{\Sigma}^1_n(X)$.

Let $x \in f^{-1}(F) = \pi_X(A)$. Since D is dense in A there exist $x_n \in \pi_X(D)$, $n \in \mathbb{N}$, such that $(x_n, f(x_n)) \to (x, f(x))$. Clearly, $\{x\} \cup \{x_n;\ n \in \mathbb{N}\} \in \mathcal{V}$. Thus, $x \in \bigcup \mathcal{V}$.

Let $K \in \mathcal{V}$. Then there exists $B \subset D$ such that $\mathrm{graph}(f) \cap (\pi_X)^{-1}(K) = \overline{B}$. So, $\mathrm{graph}(f) \cap (\pi_X)^{-1}(K) \subset A$. Thus, $K \subset \pi_X(A) = f^{-1}(F)$. □

3.3. Proof of Corollary 2.2

Lemma 3.9. *Let X, Y be Polish spaces, $f: X \to Y$ be a function and $n \in \mathbb{N}$. Then $C(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$ if and only if $\widetilde{C}(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$.*

Proof. Let $C(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. Clearly, $\widetilde{C}(f) = C(f) \cap \mathcal{O}(X)$. By Lemma 3.1 we have $\widetilde{C}(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$.

Let $\widetilde{C}(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. By Heine theorem we have

$$C(f) = \{K \in \mathcal{K}(X);\ \mathcal{O}(K) \subset \widetilde{C}(f)\}.$$

Thus we have

$$C(f) = \{K \in \mathcal{K}(X);\ \forall L \in \mathcal{K}(X)\colon\ (L \notin \mathcal{O}(K) \vee L \in \widetilde{C}(f))\}.$$

So, $C(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. $\square$

Proof of Corollary 2.2. Let $\widetilde{C}(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. By Lemma 3.9 we have $C(f) \in \boldsymbol{\Pi}^1_n(\mathcal{K}(X))$. Using [6, Theorem 2.5(ii)] we have f is $\boldsymbol{\Delta}^1_n$-measurable. By Theorem 2.1 we have $\widetilde{C}(f) \in \boldsymbol{\Delta}^1_n(\mathcal{K}(X))$. The converse implication is trivial. $\square$

3.4. Proof of Theorem 2.4

Definition 3.10. Let X, Y be Polish spaces and $\Gamma \subset 2^X$. We define $\Omega_1(X)$, $\Omega_2(X)$, $\Omega(X)$, $\boldsymbol{M}_\Gamma(X,Y)$, $\boldsymbol{E}_\Gamma(X,Y)$ by

(i) $\Omega_1(X) = \{\mathcal{V} \subset 2^X;\ \mathcal{V}$ is closed under countable unions$\}$,
(ii) $\Omega_2(X) = \{\mathcal{V} \subset 2^X;\ \mathcal{V}$ is closed under finite intersections $\wedge\ (\mathcal{V} \supset \boldsymbol{\Pi}^0_1(X) \vee \mathcal{V} \supset \boldsymbol{\Sigma}^0_1(X))\}$,
(iii) $\Omega(X) = \Omega_1(X) \cap \Omega_2(X)$,
(iv) $\boldsymbol{M}_\Gamma(X,Y) = \{f \in Y^X;\ f$ is Γ-measurable$\}$,
(v) $\boldsymbol{E}_\Gamma(X,Y)$ denotes the family of all functions $f: X \to Y$ such that

$$\forall x \in X\ \forall W \in \mathcal{U}(f(x))\ \exists U \in \mathcal{U}(x)\ \exists V \in \mathcal{U}(f(x))\ \exists G \in \Gamma\colon\quad G \supset f^{-1}(V) \cap U \quad \text{and} \quad f(G) \subset W.$$

Recall that by Lebesgue–Hausdorff–Banach theorem [3, Theorem 24.3] we have $\mathcal{B}_\alpha(X,Y) = \boldsymbol{M}_{\boldsymbol{\Sigma}^0_{\alpha+1}(X)}(X,Y)$.

Lemma 3.11. *Let X, Y be Polish spaces and $\Gamma \in \Omega_1(X)$. Then we have*

(i) $\boldsymbol{E}_{\check{\Gamma}}(X,Y) \supset \boldsymbol{M}_\Gamma(X,Y)$,
(ii) $\boldsymbol{E}_\Gamma(X,Y) = \boldsymbol{M}_\Gamma(X,Y)$.

Proof. (i) Let $f \in \boldsymbol{M}_\Gamma(X,Y)$, $x \in X$, and $W \in \mathcal{U}(f(x))$ be arbitrary. We find $V \in \mathcal{U}(f(x))$ such that $\overline{V} \subset W$. Set $G := f^{-1}(\overline{V})$ and $U := X$. Then $G \in \check{\Gamma}$, $G \supset f^{-1}(V) \cap U$ and $f(G) \subset W$. Thus we have $f \in \boldsymbol{E}_{\check{\Gamma}}(X,Y)$.

(ii) The inclusion $\boldsymbol{E}_\Gamma(X,Y) \supset \boldsymbol{M}_\Gamma(X,Y)$ can be proved similarly to (i) by setting $V := W$, $G := f^{-1}(V)$ and $U := X$.

Now we prove $\boldsymbol{E}_\Gamma(X,Y) \subset \boldsymbol{M}_\Gamma(X,Y)$. Let $f \in \boldsymbol{E}_\Gamma(X,Y)$ and an open set $W \subset Y$ be arbitrary. Let $\{U_n;\ n \in \mathbb{N}\}$ and $\{V_n;\ n \in \mathbb{N}\}$ be countable open bases of X and Y respectively. We set

$$\mathcal{M} := \{(n,m) \in \mathbb{N}^2;\ \exists G \in \Gamma\colon\ G \supset f^{-1}(V_m) \cap U_m \wedge f(G) \subset W\}.$$

For each $(n,m) \in \mathcal{M}$ we fix $G_{n,m} \in \Gamma$ satisfying $G_{n,m} \supset f^{-1}(V_m) \cap U_m$ and $f(G_{n,m}) \subset W$. Since $f \in \boldsymbol{E}_\Gamma(X,Y)$ we have that for all $x \in f^{-1}(W)$ there exists $(n_x, m_x) \in \mathcal{M}$ such that $U_{n_x} \in \mathcal{U}(x)$ and $V_{m_x} \in \mathcal{U}(f(x))$. Since $x \in G_{n_x,m_x} \subset f^{-1}(W)$ for all $x \in f^{-1}(W)$ we have $f^{-1}(W) = \bigcup_{x \in f^{-1}(W)} G_{n_x,m_x}$. Since $\Gamma \in \Omega_1$ and the set $\{G_{n_x,m_x};\ x \in f^{-1}(W)\}$ is countable, we have $f^{-1}(W) \in \Gamma$. $\square$

Lemma 3.12. *Let X, Y be Polish spaces and $1 \leqslant \alpha < \omega_1$. Then*

$$\mathcal{B}_\alpha(X,Y) \supset \mathcal{E}_{\alpha+1}(X,Y) \supset \mathcal{E}_\alpha(X,Y) \supset \bigcup_{\beta+1<\alpha} \mathcal{B}_\beta(X,Y).$$

Proof. Clearly, $\boldsymbol{\Sigma}^0_\beta(X) \in \Omega_1(X)$ for all $1 \leqslant \beta < \omega_1$. So,

$$\begin{aligned}\mathcal{B}_\alpha(X,Y) &= \boldsymbol{M}_{\boldsymbol{\Sigma}^0_{\alpha+1}(X)}(X,Y) = \boldsymbol{E}_{\boldsymbol{\Sigma}^0_{\alpha+1}(X)}(X,Y) \quad \text{(Lemma 3.11)}\\ &\supset \boldsymbol{E}_{\boldsymbol{\Pi}^0_\alpha(X)}(X,Y) = \mathcal{E}_{\alpha+1}(X,Y) \supset \mathcal{E}_\alpha(X,Y).\end{aligned}$$

Clearly,

$$\mathcal{E}_\alpha(X,Y) = \boldsymbol{E}_{\bigcup_{\beta<\alpha}\boldsymbol{\Pi}^0_\beta(X)}(X,Y) \supset \boldsymbol{E}_{\bigcup_{\beta+1<\alpha}\boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y) \supset \bigcup_{\beta+1<\alpha} \boldsymbol{E}_{\boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y).$$

By Lemma 3.11(i) we have

$$\boldsymbol{E}_{\boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y) \supset \boldsymbol{M}_{\boldsymbol{\Sigma}^0_{\beta+1}(X)}(X,Y).$$

Thus,

$$\bigcup_{\beta+1<\alpha} \boldsymbol{E}_{\boldsymbol{\Pi}^0_{\beta+1}(X)}(X,Y) \supset \bigcup_{\beta+1<\alpha} \boldsymbol{M}_{\boldsymbol{\Sigma}^0_{\beta+1}(X)}(X,Y) = \bigcup_{\beta+1<\alpha} \mathcal{B}_\beta(X,Y). \qquad \square$$

Lemma 3.13. *Let X, Y be Polish spaces, $f, g : X \to Y$ be functions, and $C(f) \supset C(g)$. Then for all $x \in X$, $V \in \mathcal{U}(f(x))$ there exist $\widetilde{V} \in \mathcal{U}(g(x))$, $\widetilde{U} \in \mathcal{U}(x)$ such that*

$$f^{-1}(V) \supset g^{-1}(\widetilde{V}) \cap \widetilde{U}. \tag{3.1}$$

Proof. Let $x \in X$ and $V \in \mathcal{U}(f(x))$ be arbitrary. Let $\{U_n;\ n \in \mathbb{N}\}$ and $\{V_n;\ n \in \mathbb{N}\}$ be decreasing sequences of open sets, which form bases of neighborhoods of x and $g(x)$ respectively. Assume towards contradiction that there are no $\widetilde{U}$ and $\widetilde{V}$ satisfying (3.1). Thus, there is a sequence $\{x_n \in X;\ n \in \mathbb{N}\}$ such that

$$x_n \in \left(g^{-1}(V_n) \cap U_n\right) \setminus f^{-1}(V). \tag{3.2}$$

So, $x_n \to x$ and $g(x_n) \to g(x)$. Thus, $\{x_n;\ n \in \mathbb{N}\} \cup \{x\} \in C(g) \subset C(f)$. By (3.2) we have $\{x_n;\ n \in \mathbb{N}\} \cup \{x\} \notin C(f)$, a contradiction. $\square$

Lemma 3.14. *Let X, Y be Polish spaces, $\Gamma \in \Omega_2(X)$, $f, g : X \to Y$ be functions and $C(f) = C(g)$. Then $f \in \boldsymbol{E}_\Gamma(X,Y)$ if and only if $g \in \boldsymbol{E}_\Gamma(X,Y)$.*

Proof. Assume that $f \in \boldsymbol{E}_\Gamma(X,Y)$ and $g \notin \boldsymbol{E}_\Gamma(X,Y)$. For every $x \in X$ let $\{U_n(x);\ n \in \mathbb{N}\}$, $\{V_n(x);\ n \in \mathbb{N}\}$ and $\{W_n(x);\ n \in \mathbb{N}\}$ be decreasing sequences of open sets, which form bases of neighborhoods of x, $f(x)$ and $g(x)$ respectively. Let $\{G^\gamma(x);\ \gamma \in \Gamma_n(x)\}$ be the family of all $G \in \Gamma$ satisfying $G \supset g^{-1}(W_n(x)) \cap U_n(x)$. Since $g \notin \boldsymbol{E}_\Gamma(X,Y)$ we have that there exist $x \in X$ and $W \in \mathcal{U}(g(x))$ such that for all $n \in \mathbb{N}$ and $\gamma \in \Gamma_n(x)$ there exists $x_n^\gamma \in G^\gamma(x)$ such that

$$g\left(x_n^\gamma\right) \notin W. \tag{3.3}$$

Since $f \in \boldsymbol{E}_\Gamma(X,Y)$ we have that for all $s \in \mathbb{N}$ there exist $\widetilde{U}^s \in \mathcal{U}(x)$, $\widetilde{V}^s \in \mathcal{U}(f(x))$ and $\widetilde{G}^s \in \Gamma$ such that $\widetilde{G}^s \supset f^{-1}(\widetilde{V}^s) \cap \widetilde{U}^s$ and $f(\widetilde{G}^s) \subset V_s(x)$. Set $U^s := U_s(x) \cap \widetilde{U}^s$ and $V^s := V_s(x) \cap \widetilde{V}^s$. If Γ contains open sets, then we set $G^s := U_s(x) \cap \widetilde{G}^s$. If Γ contains closed sets, then we set $G^s := \overline{U_s(x)} \cap \widetilde{G}^s$. Thus,

$$f^{-1}\left(V^s\right) \cap U^s \subset G^s \in \Gamma \tag{3.4}$$

and $f(G^s) \subset V_s(x)$. By Lemma 3.13 we have that for all $s \in \mathbb{N}$ there exists $m(s) \in \mathbb{N}$ such that

$$f^{-1}\left(V^s\right) \cap U^s \supset g^{-1}\left(W_{m(s)}(x)\right) \cap U_{m(s)}(x).$$

By (3.4) we have that there exists $\gamma_s \in \Gamma_{m(s)}(x)$ such that $G^s = G^{\gamma_s}(x)$. Set $y^s := x_{m(s)}^{\gamma_s}$ for $s \in \mathbb{N}$. Since $y_s \in G^s \subset f^{-1}(V_s(x))$ we have $y^s \to x$ and $f(y^s) \to f(x)$. Since $C(f) = C(g)$ we have $g(y^s) \to g(x)$. It contradicts (3.3). $\square$

Lemma 3.15. *Let X, Y be Polish spaces, $\Gamma \in \Omega(X)$, $f, g : X \to Y$ be functions and $C(f) = C(g)$. Then $f \in \boldsymbol{M}_\Gamma(X,Y)$ if and only if $g \in \boldsymbol{M}_\Gamma(X,Y)$.*

Proof. This follows from Lemmas 3.11(ii) and 3.14. $\square$

Proof of Theorem 2.4. (i) This follows from Lemma 3.14 and

$$\mathcal{E}_\alpha(X,Y) = \boldsymbol{E}_{\bigcup_{\beta<\alpha}\boldsymbol{\Pi}^0_\beta(X)}(X,Y),$$

$$\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta(X) \in \Omega_2(X).$$

Since $\boldsymbol{\Sigma}^0_{\alpha+1}(X), \mathcal{A} \in \Omega(X)$, (ii) and (iii) follow from Lemma 3.15. $\square$

3.5. *Proof of Theorem 2.7*

(i) This statement is similar to [2, Theorem 3]. We only replace $C(f)$ by $\widetilde{C}(f)$. Proof is also similar.

(ii) Following ideas of the proof of [2, Theorem 6] one can find sets $H_l^i \in \boldsymbol{\Pi}_\alpha^0(X)$, $i, l \in \mathbb{N}$, such that $\widetilde{C}(f) = \bigcap_{l\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \mathcal{O}(H_l^i)$. By Lemma 3.6(i) we have $\mathcal{O}(H_l^i) \in \boldsymbol{\Pi}_{\alpha+2}^0(\mathcal{K}(X))$. Consequently, $\widetilde{C}(f) \in \boldsymbol{\Pi}_{\alpha+4}^0(\mathcal{K}(X))$.

(iii) Similarly as in the previous case, we find sets $H_l^i \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}_\beta^0(X)$, $i, l \in \mathbb{N}$, such that $\widetilde{C}(f) = \bigcap_{l\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \mathcal{O}(H_l^i)$. By Lemma 3.6(i) we have $\mathcal{O}(H_l^i) \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}_\beta^0(X)$ since α is limit. Consequently, $\widetilde{C}(f) \in \boldsymbol{\Pi}_{\alpha+1}^0(\mathcal{K}(X))$.

3.6. *Proof of Theorem 2.8*

(i) By Theorem 2.7(i) we have $f \in \mathcal{E}_{\alpha+1}(X, Y)$. By Lemma 3.12 we have $f \in \mathcal{B}_\alpha(X, Y)$.

(ii) By Lemma 3.12 we have $f \in \mathcal{E}_{\alpha+2}(X, Y)$. By Theorem 2.7(ii) we have $\widetilde{C}(f) \in \boldsymbol{\Pi}_{\alpha+5}^0(\mathcal{K}(X))$.

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